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1994 J. Phys. A: Math. Gen. 27 4537

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Graded contractions of bilinear invariant forms of Lie algebra

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Received 28 February 1994

Abstract. We introduce a new construction for bilinear invariant forms on Lie algebras based on the method of graded contractions. The general method is described and the \mathbb{Z}_2 , \mathbb{Z}_3 and $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ contractions are found. The results can be applied to all Lie algebras and superalgebras (finite or infinite dimensional) which admit the chosen gradings. We consider some examples: contractions of the Killing form, toroidal contractions of $sl(3, \mathbb{C})$ and we briefly discuss the limit to new Wess–Zumino–Witten actions.

1. Introduction

Contractions of Lie algebras were introduced into physics forty years ago by Wigner and Inönü in order to obtain a formal method of passing from the Poincaré group to the Galilei group [1]. In general, contractions consist of the introduction of parameters into the basis of a Lie algebra such that, for some singular value of these parameters, we get a different (i.e. non-isomorphic) algebra. The interest in this method for physics stems from the fact that it relates different symmetry groups from which the *contracted* group can be understood as being an ‘approximation’ of the ‘exact’ or non-contracted group. (A discussion of the concept of contraction and some generalizations of the Wigner–Inönü method are given in [2].)

A different approach has been developed recently which is based on the preservation of some *grading* of the Lie algebra through the contraction procedure. The general method of so-called *graded contractions* is presented in [3]. Clearly, as a Lie algebra is contracted, many related objects are deformed as well, for example their representations, the tensor product of their representations, their Casimir operators, etc. The methods of graded contractions for the representations (which contain the contractions of algebras as a particular case) and their Casimir operators can be found in [4] and [5], respectively. However, contractions of bilinear invariant forms have not been considered within that framework so that one is restricted to the Killing form of the contracted algebras. The present method allows one to enlarge the possibilities and, in some sense, completes the general theory of graded contractions.

In this paper, we use the concept of graded contractions of Lie algebras to construct new symmetric bilinear invariant forms of Lie algebras (in general, non-semisimple) starting from the known bilinear form of a non-contracted algebra. Even within a single Lie algebra, one can obtain different bilinear forms starting from one bilinear invariant form. As for the method of graded contractions of algebras, the present procedure is very general and can be applied to any Lie algebra or superalgebra of finite or infinite dimension. In addition, the results are universal in the sense that, given a grading group (which is Abelian for our purpose), the problem is solved simultaneously for all Lie algebras of any type and

dimension which admit the chosen grading. The construction is presented in the next section. In section 3, we find the general bilinear invariant form obtained from preserving $\mathbb{Z}_2, \mathbb{Z}_3$ and $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ gradings. In section 4 we present some examples: contractions of the Killing form, toroidal contractions of $sl(3, \mathbb{C})$ (or, more precisely, of its complexification A_2) and the ‘deformation’ of Wess–Zumino–Witten (WZW) actions (defined over some group manifold) into modified WZW actions defined over a contracted group.

2. Description of the method

Consider a Lie algebra g (corresponding to a Lie group G) defined over the field $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$. A symmetric bilinear invariant form Ω on g is a mapping

$$\Omega : g \times g \rightarrow \mathbb{K} \tag{2.1}$$

which satisfies the symmetry property

$$\Omega(X, Y) = \Omega(Y, X) \tag{2.2}$$

and is G -invariant

$$\Omega(\vartheta X \vartheta^{-1}, \vartheta Y \vartheta^{-1}) = \Omega(X, Y) \tag{2.3}$$

with $X, Y \in g$ and $\vartheta \in G$.

In terms of the element $Z \in g$ corresponding to ϑ , (2.3) is equivalent to

$$\Omega(X, [Z, Y]) + \Omega(Y, [Z, X]) = 0. \tag{2.4}$$

Given a basis $\{X_1, \dots, X_{\dim\{g\}}\}$ of g such that $[X_A, X_B] = f_{AB}^C X_C$, the bilinear form is often written as a matrix $\Omega_{AB} \equiv \Omega(X_A, X_B)$ which is symmetric (from (2.2)) and satisfies $f_{AB}^D \Omega_{CD} + f_{AC}^D \Omega_{BD} = 0$ (from (2.4)).

Let us recall some definitions and properties that we need from the theory of graded contractions of Lie algebras (more details are in [3,4]). A Γ grading of a Lie algebra g consists of the decomposition

$$g = \bigoplus_{\mu \in \Gamma} g_\mu \tag{2.5}$$

into eigenspaces under the action of a set of automorphisms of finite order on g (for our purposes we can take Γ to be an Abelian finite group). Each of these automorphisms provides g with a grading by a cyclic group having the same order. Thus, the grading group Γ is the tensor product of all the cyclic groups associated with every automorphism: $\Gamma = \otimes \mathbb{Z}_N$.

The commutators in g inherit a grading structure from the automorphisms of finite order, i.e. if $X \in g_\mu$ and $Y \in g_\nu$, then

$$[X, Y] = Z \in g_{\mu+\nu} \tag{2.6}$$

where Z can be equal to zero. (The addition $\mu + \nu$ denotes the product of elements μ and ν in the grading group Γ .) This grading structure can be written symbolically

$$0 \neq [g_\mu, g_\nu] \subseteq g_{\mu+\nu} \quad \mu, \nu, \mu + \nu \in \Gamma. \tag{2.7}$$

A graded contraction g^ε of g is defined by modifying the commutators

$$[g_\mu, g_\nu]_\varepsilon \equiv \varepsilon_{\mu,\nu}[g_\mu, g_\nu] \subseteq \varepsilon_{\mu,\nu}g_{\mu+\nu} \tag{2.8}$$

where $\varepsilon \in \mathbb{K}$. We have kept the ε parameter in the right-hand side of (2.8) in order to show that if $\varepsilon = 0$ then the commutator of an element of g_μ with an element of g_ν vanishes. From the definition (2.8), we see that the ε parameters are symmetric (i.e. $\varepsilon_{\mu,\nu} = \varepsilon_{\nu,\mu}$) and by enforcing the Jacobi identities on the new commutators one finds that the ε parameters must satisfy the *contraction relations*

$$\varepsilon_{\mu,\nu}\varepsilon_{\mu+\nu,\lambda} = \varepsilon_{\nu,\lambda}\varepsilon_{\nu+\lambda,\mu} \tag{2.9}$$

for all values of the indices. A solution ε of (2.9) defines a contraction, i.e. a new Lie algebra.

Now we define the contractions of the bilinear invariant form Ω . Using the same notation as in (2.7), we define the *contracted symmetric bilinear invariant form* Ω^γ as

$$\Omega^\gamma(g_\mu, g_\nu) \equiv \gamma_{\mu,\nu}\Omega(g_\mu, g_\nu) \tag{2.10}$$

with $\gamma_{\mu,\nu} \in \mathbb{K}$ and $g_\mu(g_\nu)$ representing any element $X \in g_\mu(g_\nu)$.

From the properties of symmetry and invariance, and given a contraction matrix ε (i.e. a contracted algebra), we get the following restrictions upon γ :

$$\gamma_{\mu,\nu} = \gamma_{\nu,\mu} \tag{2.11}$$

and

$$\varepsilon_{\lambda,\mu}\gamma_{\lambda+\mu,\nu} = \varepsilon_{\lambda,\nu}\gamma_{\lambda+\nu,\mu}. \tag{2.12}$$

Proof. The relation (2.11) follows from equations (2.2) and (2.10).

The condition (2.12) is obtained from substituting (2.10) into (2.4)

$$\begin{aligned} \Omega^\gamma(g_\mu, [g_\lambda, g_\nu]_\varepsilon) + \Omega^\gamma(g_\nu, [g_\lambda, g_\mu]_\varepsilon) &= 0 \\ \Omega^\gamma(g_\mu, \varepsilon_{\lambda,\nu}[g_\lambda, g_\nu]) + \Omega^\gamma(g_\nu, \varepsilon_{\lambda,\mu}[g_\lambda, g_\mu]) &= 0 \\ \varepsilon_{\lambda,\nu}\gamma_{\mu,\lambda+\nu}\Omega(g_\mu, [g_\lambda, g_\nu]) + \varepsilon_{\lambda,\mu}\gamma_{\nu,\lambda+\mu}\Omega(g_\nu, [g_\lambda, g_\mu]) &= 0. \end{aligned}$$

By comparing the last line with the invariance condition relation

$$\Omega(g_\mu, [g_\lambda, g_\nu]) + \Omega(g_\nu, [g_\lambda, g_\mu]) = 0$$

before contraction, we see that one must have $\varepsilon_{\lambda,\nu}\gamma_{\mu,\lambda+\nu} = \varepsilon_{\lambda,\mu}\gamma_{\nu,\lambda+\mu}$. The form (2.12) is obtained by using the symmetry properties of ε and γ . \square

The solutions of (2.11) and (2.12) provide new invariant bilinear forms obtained by substituting the γ parameters back into (2.10). As for the contractions of algebras and representations, there is a trivial contracted bilinear form for which all γ s are equal to zero. However, the solution with all the γ s equal to one is not trivial. As explained in [3], the equations in (2.9) which contain an undefined ε (when the corresponding commutator has already vanished in the non-contracted algebra) do not appear in the system of equations

to be solved. The same occurs for the γ s. If μ and ν are such that $\Omega(g_\mu, g_\nu) = 0$ then the equations containing the corresponding $\gamma_{\mu,\nu}$ must be removed from the system (2.12). If it does not happen, we say that this grading is *generic* regarding the invariant bilinear form. Note that definitions (2.8) and (2.10) also allow one to consider *deformations*, i.e. processes which occur after some commutators or bilinear forms—initially zero—become non-trivial but which nevertheless satisfy the grading property. We shall not consider this type of process below.

Note also that here, in general, the ‘composition’ $(\gamma_1 \bullet \gamma_2)_{\mu,\nu} \equiv (\gamma_1)_{\mu,\nu}(\gamma_2)_{\mu,\nu}$ (without summation over repeated indices) of solution matrices, which we have defined for the contractions of algebras in equations (2.8) and (2.9) of [3], does not yield a new solution. Thus, given two solutions of (2.12), γ_1 and γ_2 , their composition will not be a solution unless extremely particular conditions on the ε s are satisfied. The same applies for the ‘normalization’ of the γ s because it is related to the normalization of the ε s. If we perform a change of basis $g_\mu \rightarrow g'_\mu = a_\mu g_\mu$ then the invariant bilinear form becomes $\Omega^y(g'_\mu, g'_\nu) = \gamma_{\mu,\nu}/(a_\mu a_\nu)\Omega(g_\mu, g_\nu)$. Unless we consider very particular values of γ s and ε s, it will not be possible to satisfy $\gamma_{\mu,\nu}/(a_\mu a_\nu) = 1$ for all γ s. Unlike the contractions of representations, where we can normalize the parameters ψ (see [4]) by changing the basis of the representation vector space independently of the basis of the algebra, here the normalizations of ε and γ are interdependent.

3. $\mathbb{Z}_{1|b|2}$, \mathbb{Z}_3 and $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ contractions

In this section, we find the contractions of the invariant bilinear forms, using the results of [3], for Lie algebras over the field of *complex* numbers. As in [3], we consider the contractions with gradings of *generic* type. We have not normalized the γ s in order to display the possible zero parameters. (Note that the ε s of the present paper are the γ s in [3].)

3.1. \mathbb{Z}_2 contractions

The grading group consists of two elements $\Gamma = \{0, 1\}$ with the product (denoted additively)

$$0 + 0 = 0 \quad 0 + 1 = 1 + 0 = 1 \quad 1 + 1 = 0.$$

There are three independent γ parameters: $\gamma_{0,0}$, $\gamma_{0,1}$ ($= \gamma_{1,0}$) and $\gamma_{1,1}$. We shall cast them into a matrix form

$$\gamma = \begin{pmatrix} \gamma_{0,0} & \gamma_{0,1} \\ \gamma_{0,1} & \gamma_{1,1} \end{pmatrix}$$

although it should not be considered formally as a matrix, since none of the properties of matrices—multiplication, inverse, determinant, etc—is shared by the present objects. (Hereafter, we write only the upper triangular elements of γ and ε , remembering that they are symmetric.)

The condition (2.12) gives, for \mathbb{Z}_2 ,

$$\varepsilon_{0,0}\gamma_{0,1} = \varepsilon_{0,1}\gamma_{0,1} \quad \varepsilon_{0,1}\gamma_{1,1} = \varepsilon_{1,1}\gamma_{0,0}. \tag{3.1}$$

For the trivial contraction

$$\varepsilon = \begin{pmatrix} \varepsilon_{0,0} & \varepsilon_{0,1} \\ & \varepsilon_{1,1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} = (1)$$

the relations (3.1) become $\gamma_{0,1} = \gamma_{0,1}$ and $\gamma_{0,0} = \gamma_{1,1}$ (which imposes no further restriction upon $\gamma_{0,1}$). For the other trivial contraction

$$\varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = (0)$$

there are no restrictions on γ which are thus completely free. If $\varepsilon_{0,0} \neq \varepsilon_{0,1}$ then $\gamma_{0,1}$ must vanish. This happens for the Wigner–Inönü-like contraction matrix

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In summary, we find the following contractions:

$$\begin{aligned} \varepsilon &= \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} & \gamma &= \begin{pmatrix} a & b \\ & a \end{pmatrix} \\ \varepsilon &= \begin{pmatrix} 0 & 0 \\ & 0 \end{pmatrix} & \gamma &= \begin{pmatrix} a & b \\ & c \end{pmatrix} \\ \varepsilon &= \begin{pmatrix} 1 & 0 \\ & 0 \end{pmatrix} & \gamma &= \begin{pmatrix} a & 0 \\ & b \end{pmatrix} \\ \varepsilon &= \begin{pmatrix} 0 & 0 \\ & 1 \end{pmatrix} & \gamma &= \begin{pmatrix} 0 & a \\ & b \end{pmatrix} \\ \varepsilon &= \begin{pmatrix} 1 & 1 \\ & 0 \end{pmatrix} & \gamma &= \begin{pmatrix} a & b \\ & 0 \end{pmatrix} \end{aligned} \tag{3.2}$$

where a, b and c are arbitrary (possibly zero) complex numbers.

3.2. \mathbb{Z}_3 contractions

The \mathbb{Z}_3 contractions (\mathbb{Z}_3 consists of three elements 0, 1, 2 on which the product is the usual addition modulo 3) are determined by six parameters (cast into matrix form)

$$\gamma = \begin{pmatrix} \gamma_{0,0} & \gamma_{0,1} & \gamma_{0,2} \\ & \gamma_{1,1} & \gamma_{1,2} \\ & & \gamma_{2,2} \end{pmatrix}.$$

In terms of these parameters, the restrictions (2.12) are

$$\begin{aligned} \varepsilon_{0,0}\gamma_{0,1} = \varepsilon_{0,1}\gamma_{0,1} & \quad \varepsilon_{1,1}\gamma_{2,2} = \varepsilon_{1,2}\gamma_{0,1} & \quad \varepsilon_{0,0}\gamma_{0,2} = \varepsilon_{0,2}\gamma_{0,2} \\ \varepsilon_{0,2}\gamma_{1,2} = \varepsilon_{1,2}\gamma_{0,0} & \quad \varepsilon_{0,1}\gamma_{1,2} = \varepsilon_{0,2}\gamma_{1,2} & \quad \varepsilon_{0,2}\gamma_{2,2} = \varepsilon_{2,2}\gamma_{0,1} \\ \varepsilon_{0,1}\gamma_{1,1} = \varepsilon_{1,1}\gamma_{0,2} & \quad \varepsilon_{1,2}\gamma_{0,2} = \varepsilon_{2,2}\gamma_{1,1} & \quad \varepsilon_{0,1}\gamma_{1,2} = \varepsilon_{1,2}\gamma_{0,0} \end{aligned} \tag{3.3}$$

And the solutions are (with the ε s given in section 4 of [3]):

$$\begin{aligned}
 \gamma(\varepsilon^I) &= \begin{pmatrix} a & b & c \\ & c & 0 \\ & & 0 \end{pmatrix} & \gamma(\varepsilon^{II}) &= \begin{pmatrix} a & b & c \\ & 0 & 0 \\ & & b \end{pmatrix} \\
 \gamma(\varepsilon^{III}) &= \begin{pmatrix} a & b & c \\ & 0 & 0 \\ & & 0 \end{pmatrix} & \gamma(\varepsilon^{IV}) &= \begin{pmatrix} a & 0 & 0 \\ & b & c \\ & & 0 \end{pmatrix} \\
 \gamma(\varepsilon^V) &= \begin{pmatrix} a & 0 & 0 \\ & 0 & b \\ & & c \end{pmatrix} & \gamma(\varepsilon^{VI}) &= \begin{pmatrix} a & 0 & 0 \\ & b & c \\ & & d \end{pmatrix} \\
 \gamma(\varepsilon^{VII}) &= \begin{pmatrix} a & b & 0 \\ & 0 & 0 \\ & & c \end{pmatrix} & \gamma(\varepsilon^{VIII}) &= \begin{pmatrix} a & 0 & b \\ & c & 0 \\ & & 0 \end{pmatrix} \\
 \gamma(\varepsilon^{IX}) &= \begin{pmatrix} 0 & a & 0 \\ & b & c \\ & & a \end{pmatrix} & \gamma(\varepsilon^X) &= \begin{pmatrix} 0 & 0 & a \\ & a & b \\ & & c \end{pmatrix} \\
 \gamma(\varepsilon^{XI}) &= \begin{pmatrix} 0 & 0 & 0 \\ & a & b \\ & & c \end{pmatrix} & \gamma(\varepsilon^{XII}) &= \begin{pmatrix} a & b & 0 \\ & c & d \\ & & 0 \end{pmatrix} \\
 \gamma(\varepsilon^{XIII}) &= \begin{pmatrix} a & 0 & b \\ & 0 & c \\ & & d \end{pmatrix} & \gamma(\varepsilon = (1)) &= \begin{pmatrix} a & b & c \\ & c & a \\ & & b \end{pmatrix}
 \end{aligned} \tag{3.4}$$

where a, b, c and d are arbitrary. The matrix $\varepsilon = (1)$ has all its entries equal to 1 (it is not the identity matrix). Obviously, the γ s corresponding to $\varepsilon = (0)$ are free.

3.3. $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ contractions.

The grading group consists of four elements that we name $a = 00, b = 01, c = 10$ and $d = 11$ so that their product is $a + k = k, 2k = a$ ($k = a, \dots, d$) and $b + c = d, c + d = b$ and $b + d = c$. In the generic case (i.e. when all the ε s are defined), the equations (2.12) are

$$\begin{aligned}
 \varepsilon_{a,a}\gamma_{a,k} &= \varepsilon_{a,k}\gamma_{a,k} & \varepsilon_{a,k}\gamma_{k,k} &= \varepsilon_{k,k}\gamma_{a,a} \\
 \varepsilon_{a,b}\gamma_{b,c} &= \varepsilon_{a,c}\gamma_{b,c} = \varepsilon_{b,c}\gamma_{a,d} \\
 \varepsilon_{a,b}\gamma_{b,d} &= \varepsilon_{a,d}\gamma_{b,d} = \varepsilon_{b,d}\gamma_{a,c} \\
 \varepsilon_{a,c}\gamma_{c,d} &= \varepsilon_{a,d}\gamma_{c,d} = \varepsilon_{c,d}\gamma_{a,b} \\
 \varepsilon_{b,c}\gamma_{d,d} &= \varepsilon_{b,d}\gamma_{c,c} = \varepsilon_{c,d}\gamma_{b,b} \\
 \varepsilon_{b,b}\gamma_{a,c} &= \varepsilon_{b,c}\gamma_{b,d} & \varepsilon_{b,b}\gamma_{a,d} &= \varepsilon_{b,d}\gamma_{b,c} \\
 \varepsilon_{b,c}\gamma_{c,d} &= \varepsilon_{c,c}\gamma_{a,b} & \varepsilon_{c,c}\gamma_{a,d} &= \varepsilon_{c,d}\gamma_{b,c} \\
 \varepsilon_{c,d}\gamma_{b,d} &= \varepsilon_{d,d}\gamma_{a,c} & \varepsilon_{b,d}\gamma_{c,d} &= \varepsilon_{d,d}\gamma_{a,b}
 \end{aligned} \tag{3.5}$$

where $k = b, c, d$. The γ solutions (corresponding to the ε solutions given in [3]) are in tables 1 and 2.

Table 1. $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ contractions of invariant bilinear forms. The γ solutions corresponding to the contractions of algebras given in table 1 of [3] on which they must be superposed.

$\begin{pmatrix} a & b & c & d \\ & a & d & c \\ & & a & b \\ & & & a \end{pmatrix}$	$\begin{pmatrix} a & b & c & d \\ & 0 & d & 0 \\ & & a & b \\ & & & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a & 0 & b \\ & c & b & d \\ & & 0 & a \\ & & & e \end{pmatrix}$	$\begin{pmatrix} a & 0 & b & 0 \\ & c & 0 & d \\ & & a & 0 \\ & & & e \end{pmatrix}$
$\begin{pmatrix} a & b & c & d \\ & a & d & c \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$	$\begin{pmatrix} a & b & c & d \\ & 0 & d & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a & 0 & b \\ & c & b & d \\ & & 0 & 0 \\ & & & e \end{pmatrix}$	$\begin{pmatrix} a & 0 & b & 0 \\ & c & 0 & d \\ & & 0 & 0 \\ & & & e \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & a & b \\ & 0 & b & a \\ & & c & d \\ & & & e \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & a & b \\ & 0 & b & 0 \\ & & c & d \\ & & & e \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & a \\ & b & c & d \\ & & e & f \\ & & & g \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & a & 0 \\ & b & c & d \\ & & e & f \\ & & & g \end{pmatrix}$
$\begin{pmatrix} a & b & 0 & 0 \\ & a & 0 & 0 \\ & & c & d \\ & & & e \end{pmatrix}$	$\begin{pmatrix} a & b & 0 & 0 \\ & 0 & 0 & 0 \\ & & c & d \\ & & & e \end{pmatrix}$	$\begin{pmatrix} 0 & a & 0 & 0 \\ & b & c & d \\ & & e & f \\ & & & g \end{pmatrix}$	$\begin{pmatrix} a & 0 & 0 & 0 \\ & b & c & d \\ & & e & f \\ & & & g \end{pmatrix}$

4. Examples

4.1. Contractions of Killing form

First, consider the algebra A_1 (the complexification of $sl(2, \mathbb{C})$) with basis elements $\{X, Y, Z\}$ and commutation relations

$$[X, Y] = Z \quad [Y, Z] = X \quad [Z, X] = Y.$$

From these we find that the Killing form $\Omega^{\text{Killing}}(\cdot, \cdot) \equiv \text{Tr}(\text{ad}(\cdot) \text{ad}(\cdot))$ is equal to

$$\Omega^{\text{Killing}} = \text{diag}(-2, -2, -2). \tag{4.1}$$

Now consider the $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ grading

$$L_b = \mathbb{C}X \quad L_c = \mathbb{C}Y \quad L_d = \mathbb{C}Z \tag{4.2}$$

where we use the same notation for the grading-group elements as in subsection 3.3. Following (2.8), the commutators are modified to

$$[X, Y]_e = \varepsilon_{b,c}Z \quad [Y, Z]_e = \varepsilon_{c,d}X \quad [Z, X]_e = \varepsilon_{b,d}Y \tag{4.3}$$

so that the Killing form that we get from the adjoint representation is

$$\Omega^{\text{Killing}, \varepsilon} = -2 \text{diag}(\varepsilon_{b,c}\varepsilon_{b,d}, \varepsilon_{b,c}\varepsilon_{c,d}, \varepsilon_{b,d}\varepsilon_{c,d}). \tag{4.4}$$

Now if we apply the definition (2.10) for the grading (4.2), the form (4.1) becomes

$$\Omega^{\text{Killing}, \gamma} = -2 \text{diag}(\gamma_{b,b}, \gamma_{c,c}, \gamma_{d,d}) \tag{4.5}$$

Table 2. $Z_2 \otimes Z_2$ contractions of invariant bilinear forms. The γ solutions corresponding to the contractions of algebras given in table 2 of [3] on which they must be superposed.

$\begin{pmatrix} a & b & c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$	$\begin{pmatrix} a & 0 & 0 & b \\ c & d & 0 & 0 \\ e & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & b & 0 & c \\ 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 & b & c \\ d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & b & c & 0 \\ d & e & 0 & 0 \\ f & 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a & b & 0 \\ c & d & b & 0 \\ e & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & b & 0 & c \\ d & 0 & 0 & e \\ 0 & 0 & 0 & f \end{pmatrix}$	$\begin{pmatrix} a & 0 & 0 & 0 \\ b & 0 & c & d \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & b & c & e \\ d & e & f & \end{pmatrix}$
$\begin{pmatrix} 0 & a & b & 0 \\ c & d & 0 & 0 \\ e & a & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a & b & 0 \\ c & d & b & a \\ e & a & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 & b & c \\ 0 & 0 & 0 & 0 \\ d & e & f & \end{pmatrix}$	$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b & c & d \end{pmatrix}$	$\begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 & 0 & b \\ c & d & 0 & 0 \\ e & 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & b & c & e \\ d & e & f & \end{pmatrix}$	$\begin{pmatrix} 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & 0 & a & e \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & b & c & e \\ d & e & f & \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ c & d & e \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & a \\ c & d & e \end{pmatrix}$	$\begin{pmatrix} a & b & c & d \\ 0 & 0 & 0 & c \\ 0 & 0 & b & a \end{pmatrix}$

which is to be compared to (4.4). This is achieved by noting that the equations (2.12) which are relevant here (i.e. in which ε and γ do not contain a as an index) are

$$\varepsilon_{b,c}\gamma_{d,d} = \varepsilon_{c,d}\gamma_{b,b} = \varepsilon_{b,d}\gamma_{c,c} \tag{4.6}$$

for which

$$\gamma_{b,b} = \varepsilon_{b,c}\varepsilon_{b,d} \quad \gamma_{c,c} = \varepsilon_{b,c}\varepsilon_{c,d} \quad \gamma_{d,d} = \varepsilon_{b,d}\varepsilon_{c,d} \tag{4.7}$$

is a possible solution. By comparing (4.5) with (4.4), we see that the form obtained by graded contractions of A_1 (i.e. by the introduction of ε parameters into the commutators) can be compatible with the γ -contracted form obtained by the present procedure, i.e. using (2.10).

A similar compatibility exists in any general Lie algebra g with basis $\{X_1, \dots, X_{\dim[g]}\}$ such that

$$[X_A, X_B] = f_{AB}^C X_C = \text{ad}(X_A)_{CB} X_C. \tag{4.8}$$

Suppose that X_A, X_B belong to the Γ -grading subspaces g_μ, g_ν , respectively, so that $X_C \in g_{\mu+\nu}$. Then the Killing form is

$$\begin{aligned} \Omega^{\text{Killing}}(X_{\mu A}, X_{\nu B}) &= \text{Tr}[\text{ad}(X_{\mu A}) \text{ad}(X_{\nu B})] \\ &= \sum_{C,D} \text{ad}(X_{\mu A})_{(\mu+\sigma)C;\sigma D} \text{ad}(X_{\nu B})_{(\mu+\nu+\sigma)D;(\mu+\sigma)C} \end{aligned} \tag{4.9}$$

from which we see that μ must be the Γ inverse of ν . (We have written the grading indices along with the algebra indices.)

By introducing ε parameters, as in (2.8), this becomes

$$\Omega^{\text{Killing},\varepsilon}(X_{\mu A}, X_{\nu B}) = \sum_{C,D} \varepsilon_{\mu,\sigma} \varepsilon_{\nu,\mu+\sigma} \text{ad}(X_{\mu A})_{(\mu+\sigma)C;\sigma D} \text{ad}(X_{\nu B})_{\sigma D,(\mu+\sigma)C} \tag{4.10}$$

where the ε 's are relevant if the corresponding commutators are defined. One can compare (4.10) to the form obtained via our definition (2.10), i.e.

$$\Omega^{\text{Killing},\gamma}(X_{\mu A}, X_{\nu B}) = \gamma_{\mu,\nu} \Omega^{\text{Killing}}(X_{\mu A}, X_{\nu B}) \tag{4.11}$$

where $\mu + \nu = 0$ (the identity element in Γ) if the algebra is such that ε^2 in (4.10) can be removed by factorizing. Then one can identify

$$\gamma_{\mu,\nu} = \sum_{\sigma} \varepsilon_{\mu,\sigma} \varepsilon_{\nu,\mu+\sigma} \tag{4.12}$$

where the sum is over the σ indices such that the ε s are defined and with $\mu + \nu = 0$. In the particular case of A_1 , (4.12) is just (4.7).

4.2. Toroidal contractions of A_2

There are four different fine gradings of A_2 (complexification of the simple Lie algebra $sl(3, \mathbb{C})$) [6, 7], one of which is the *toroidal* (or *Cartan*) grading

$$A_2 = \bigoplus_{\mu=-3}^3 \mathfrak{g}_\mu$$

with

$$\begin{aligned} g_1 &= \mathbb{C}e_\alpha, g_{-1} = \mathbb{C}e_{-\alpha} \\ g_2 &= \mathbb{C}e_\beta, g_{-2} = \mathbb{C}e_{-\beta} \\ g_3 &= \mathbb{C}e_{\alpha+\beta}, g_{-3} = \mathbb{C}e_{-(\alpha+\beta)} \\ g_0 &= \mathbb{C}h_\alpha + \mathbb{C}h_\beta. \end{aligned} \tag{4.13}$$

for which the grading group is \mathbb{Z}_7 .

Consider the following invariant bilinear form on A_2 :

$$\begin{aligned} \Omega(a_\delta, a_{-\delta}) &= \Omega_\delta \\ \Omega(h_{\alpha_i}, h_{\alpha_j}) &= \langle \alpha_i | \alpha_j \rangle \end{aligned} \tag{4.14}$$

where $i, j = 1, 2$, δ is any root of A_2 and all other elements of Ω are zero. The only relevant γ parameters are $\gamma_{0,0}$ and $\gamma_{\mu,-\mu}$.

The grading (4.13) is such that the relevant ε parameters are

$$\varepsilon_{1,2}, \varepsilon_{1,-3}, \varepsilon_{1,-1}, \varepsilon_{-2,-1}, \varepsilon_{2,-3}, \varepsilon_{2,-2}, \varepsilon_{3,-3}, \varepsilon_{3,-2}, \varepsilon_{3,-1}, \varepsilon_{0,\mu} \quad (\mu = -3, \dots, 3).$$

From (2.12), we get the following relations:

$$\varepsilon_{\mu,-\mu} \gamma_{0,0} = \varepsilon_{0,\mu} \gamma_{\mu,-\mu} = \varepsilon_{0,-\mu} \gamma_{\mu,-\mu} \tag{4.15}$$

for $\mu = 1, 2, 3$. The possible ε parameters have been found and the associated algebras identified in [8]. The γ solutions are to be substituted into

$$\begin{aligned} \Omega^\gamma(e_\alpha, e_{-\alpha}) &= \gamma_{1,-1} \Omega_\alpha \\ \Omega^\gamma(e_\beta, e_{-\beta}) &= \gamma_{2,-2} \Omega_\beta \\ \Omega^\gamma(e_{\alpha+\beta}, e_{-(\alpha+\beta)}) &= \gamma_{3,-3} \Omega_{\alpha+\beta} \\ \Omega^\gamma(h_{\alpha_i}, h_{\alpha_j}) &= \gamma_{0,0} \langle \alpha_i | \alpha_j \rangle. \end{aligned} \tag{4.16}$$

The method can be applied readily to Kac–Moody algebras. Some examples of graded contractions of Kac–Moody algebras are given in [3]. (The particular case of Wigner–Inönü contractions of Kac–Moody and Virasoro algebras has been studied explicitly in [9] and [10], respectively.)

4.3. Application to WZW models

In this subsection, we exploit the fact that, starting from a Lie algebra g with invariant bilinear form Ω , one may define a WZW action on the corresponding group manifold [11]. Suppose that the connected group G , which corresponds to the Lie algebra g , is simply connected. Consider a three-manifold B with boundary $\Sigma = \partial B$ and ϑ (a map from Σ to G) extended to a map from B to G (see [11]). The WZW action on a Riemann surface Σ is (see [12] for further details)

$$S_{\text{WZW}}(\vartheta) = \frac{1}{4\pi} \int_{\Sigma} d^2y \Omega_{AB} A_a^A A^{Ba} + \frac{i}{12\pi} \int_B d^3y \epsilon_{abc} A^{Aa} A^{Bb} A^{Cc} \Omega_{CD} f_{AB}^D \tag{4.17}$$

where a, b, c represent indices over Σ and B and $\vartheta^{-1} \partial_a \vartheta = A_a^A X_A$ (the X s being basis elements of the Lie algebra g).

Now, if we consider a Γ grading of g such that $X_A \in g_A, X_B \in g_B$ etc (A, B, \dots denote the grading labels of the subspace to which X_A, X_B, \dots belong, respectively), then we get a new WZW action

$$S_{\text{WZW}}^{\epsilon, \gamma}(\vartheta) = \frac{1}{4\pi} \int_{\Sigma} d^2y \gamma_{A,B} \Omega_{AB} A_a^A A^{Ba} + \frac{i}{12\pi} \int_B d^3y \epsilon_{abc} A^{Aa} A^{Bb} A^{Cc} \gamma_{C,D} \Omega_{CD} \epsilon_{A,B} f_{AB}^D. \tag{4.18}$$

The invariant bilinear form Ω^γ must be non-degenerate, i.e. the inverse $\Omega^{\gamma, AB}$ of Ω_{AB}^γ must be defined such that $\Omega^{\gamma, AB} \Omega_{BC}^\gamma = \delta_C^A$. The construction of a WZW model based on a non-semisimple group has been discussed recently in [12] and [13] for ungauged and gauged models, respectively.

From this point of view, we see that the contraction procedure may provide a new geometry (in particular, a new spacetime) by starting from the geometry associated with the non-contracted WZW model and by ‘deforming’ it compatibly with the deformation of the algebra. The concept of contraction has not been mentioned explicitly in [12, 13] although the invariant bilinear form used therein was obtained from a contraction procedure [14]. To our knowledge, contraction methods have been used explicitly for the first time in [15, 16]. The contraction used in [15] leaves the contracted algebras with the particular \mathbb{Z}_3 structure

$$\epsilon = \begin{pmatrix} 1 & 1 & 1 \\ & 1 & 0 \\ & & 0 \end{pmatrix}$$

which suggests a generalization along our point of view. A detailed discussion about this topic is postponed to a future publication.

The deformation of a geometry into another one, using the concept of contraction, has been studied in [17] (although using a completely different approach). Group contractions, interpreted as quasi-catastrophical connections between different geometries or topological fluctuations in spacetime, have been considered in [18], establishing a relation between different models of the universe.

Acknowledgments

The author is indebted to Drs R C Myers and R T Sharp for reading the manuscript and to the referees for useful comments. This work has been supported by the Natural Sciences and Engineering Research Council of Canada.

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